A skew product map with a non-contracting iterated monodromy group

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The map

Consider the map \( F(z, p) = \left( \frac{z^2 - p^2}{z^2 - 1}, p^2 \right). \)
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It is *post-critically finite* with the post-critical set $P_F$ consisting of the lines
\[
\{z = 0\} \leftrightarrow \{z = p\}, \quad \{z = \infty\} \leftrightarrow \{z = 1\}, \quad \{p = 0\}, \quad \{p = \infty\}.
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\{ $z = 0$ $\leftrightarrow$ $z = p$, $z = \infty$ $\leftrightarrow$ $z = 1$, $p = 0$, $p = \infty$ \}.

The map $F : F^{-1}(\mathbb{C}^2 \setminus P_F) \longrightarrow \mathbb{C}^2 \setminus P_F$ is a covering map of topological degree 4 of a space by its subset.
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Iterated monodromy group

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\[
\gamma \cdot x_z = x_{\sigma(z)} \cdot \gamma_z
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for some permutation \(\sigma\) of \(f^{-1}(t)\) and a function \((\gamma_z)_{z \in f^{-1}(t)} \in \pi_1(M, t)^{f^{-1}(t)}\).
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$$\gamma \cdot x_z = x_{\sigma(z)} \cdot \gamma_z$$

for some permutation $\sigma$ of $f^{-1}(t)$ and a function $(\gamma_z)_{z \in f^{-1}(t)} \in \pi_1(\mathcal{M}, t)^{f^{-1}(t)}$. We get a group homomorphism $\pi_1(\mathcal{M}, t) \longrightarrow S_d \rtimes \pi_1(\mathcal{M}, t)^d$ for $d = \deg f$ called the \textit{wreath recursion}. 

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The biset is uniquely described by the associated \textit{wreath recursion}. Choose for every \( z \in f^{-1}(t) \) one connecting path \( x_z \) from \( t \) to \( z \). Then every element of the biset can be written as \( x_z \cdot \gamma \) for some \( z \in f^{-1}(t) \) and \( \gamma \in \pi_1(\mathcal{M}, t) \). In particular, for every \( \gamma \in \pi_1(\mathcal{M}, t) \) and \( z \in f^{-1}(t) \) we have

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for some permutation \( \sigma \) of \( f^{-1}(t) \) and a function \((\gamma_z)_{z \in f^{-1}(t)} \in \pi_1(\mathcal{M}, t)^{f^{-1}(t)}\). We get a group homomorphism

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for \( d = \deg f \) called the \textit{wreath recursion}. Choosing a different collection of connecting paths \( x_z \) (and different identification of \( f^{-1}(t) \) with \( \{1, 2, \ldots, d\} \)) amounts to post-composing the wreath recursion with an inner automorphism of the wreath product.
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We get a post-critically finite branched self-map $f : S^2 \to S^2$ (a Thurston map). It is obstructed (not equivalent to a rational function).
Consider the moduli space of the sphere with four marked points (the copies of 0 and \(-1\) in both planes).
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If we take the bundle over the moduli space of the corresponding complex structures on $S_2$, then the map $f$ induces the associated \textit{skew-product} map.
The bundle for the mating

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$$f_{p_1} : (\hat{C}, \infty, 1, 0, p_1) \longrightarrow (\hat{C}, 1, \infty, p_2, 0)$$

of degree 2.
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of degree 2. It has to be of the form $(z^2 + a)/(z^2 + b)$ since 0 and $\infty$ are critical points and $f_{p_1}(\infty) = 1$. 
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Let us understand the skew-product map for our case. Identify the critical point in one of the hemispheres with 0, the critical point and critical value in the second hemisphere with \( \infty \) and 1, respectively. Then the position \( p \) of the fourth point (the critical value on the first hemisphere) is a coordinate on the moduli space. We want to represent \( f \) by a rational function

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f_{p_1} : (\hat{\mathbb{C}}, \infty, 1, 0, p_1) \rightarrow (\hat{\mathbb{C}}, 1, \infty, p_2, 0)
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of degree 2. It has to be of the form \( (z^2 + a)/(z^2 + b) \) since 0 and \( \infty \) are critical points and \( f_{p_1}(\infty) = 1 \). We have \( f_{p_1}(1) = \infty \), hence \( b = -1 \). Since \( f_{p_1}(0) = p_2 \), we have \( f_{p_1}(z) = \frac{z^2 - p_2}{z^2 - 1} \). Since \( f_{p_1}(p_1) = 0 \), we have \( p_2 = p_1^2 \).

We see that \( (z, p_1) \mapsto (f_{p_1}(z), p_2) \) is \( (z, p) \mapsto \left( \frac{z^2 - p^2}{z^2 - 1}, p^2 \right) \), i.e., our map \( F \).
A skew product map with similar properties is

$$F(z, p) = \left( \frac{z^2 - (2p^2 - 1)}{z^2 - 1}, 2p^2 - 1 \right).$$
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\[ F(z, p) = \left( \frac{z^2 - (2p^2 - 1)}{z^2 - 1}, 2p^2 - 1 \right). \]

It is obtained starting from a Thurston map with the post-critical portrait

\[ \ast \implies x_1 \implies x_2 \implies x_3 \implies x_4 \implies x_3 \]
The given interpretation of the map $F$ as coming from the bundle over the moduli space can be used to compute the iterated monodromy group $\text{IMG}(F)$.
The iterated monodromy group of $F$

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\begin{align*}
a_1 & \mapsto \sigma(a_1^{-1}, b_1 a_1), & a_2 & \mapsto \sigma(a_2^{-1}, b_2 a_2), \\
b_1 & \mapsto (1, a_1), & b_2 & \mapsto (1, a_2).
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b_1 \mapsto (1, a_1), \quad b_2 \mapsto (1, a_2).
\]

Since we have to interpret this as a wreath recursion on the fundamental group of the sphere without four points, we have to impose $b_1a_1 = b_2a_2$. 
We get therefore the recursion $\phi_0$:

\[
\begin{align*}
    a_1 & \mapsto \sigma(a_1^{-1}, b_1 a_1), \\
    b_1 & \mapsto (1, a_1), \\
    b_2 & \mapsto (1, b_2^{-1} b_1 a_1).
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over the free group.
We get therefore the recursion $\phi_0$:

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over the free group. Define also the following recursion $\phi_1$:

\[
\begin{align*}
    a_1 &\mapsto \sigma(1, a_1^{-1} b_1 a_1), \\
    b_1 &\mapsto (a_1, 1), \\
    b_2 &\mapsto (1, b_2^{-1} b_1 a_1).
\end{align*}
\]
We take the generators equal to the Dehn twists

\[ a_1^T = a_1^{b_1a_1}, \quad b_1^T = b_1^{b_1a_1}, \quad b_2^T = b_2, \]

\[ T \text{ is the twist about the equator of the mating.} \]
\[ D_1 \text{ is the twist about the Thurston obstruction.} \]
We take the generators equal to the Dehn twists

\[ a_1^T = a_1 b_1 \, a_1, \quad b_1^T = b_1 b_1 a_1, \quad b_2^T = b_2, \]

and

\[ a_1^D = a_1 b_2^{-1}, \quad b_1^D = b_1, \quad b_2^D = b_2. \]
We take the generators equal to the Dehn twists

\[ a^T_1 = a_1^{b_1a_1}, \quad b^T_1 = b_1^{b_1a_1}, \quad b^T_2 = b_2, \]

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\[ a^D_1 = a_1^{b_1^{-1}b_2}, \quad b^D_1 = b_1, \quad b^D_2 = b_2. \]

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We take the generators equal to the Dehn twists

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a_1^D = a_1^{b_1^{-1} b_2}, \quad b_1^D = b_1, \quad b_2^D = b_2.
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\(T\) is the twist about the equator of the mating. \(D\) is the twist about the Thurston obstruction.
It is checked directly that $\phi_0 \circ T = \phi_1$. We have to use right action here, so we write $T \cdot \phi_0 = \phi_1$. 
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$$
\phi_1 \circ T (a_1) = \phi_1 (a_b a_1) = (\sigma(1, a_{-1} b_1 a_1)) (\sigma(1, a_1)(1, a_{-1} b_1 a_1)) = \sigma(1, b_1 a_1) = (1, b_{-1} b_1 a_1).
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It is checked directly that $\phi_0 \circ T = \phi_1$. We have to use right action here, so we write $T \cdot \phi_0 = \phi_1$. Let us compute $T \cdot \phi_1 = \phi_1 \circ T$:

$$\phi_1 \circ T(a_1) = \phi_1(a_1^{b_1a_2}) =$$
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\phi_1 \circ T(a_1) = \phi_1(a_1^{b_1a_2}) = (\sigma(1, a_1^{-1}b_1a_1))^{(a_1,1)}(1,a_1^{-1}b_1a_1) = (\sigma(1, a_1^{-1}b_1a_1))^{(1,a_1)}(1,a_1^{-1}b_1a_1) = (\sigma(1, a_1^{-1}b_1a_1))^{(1,b_1a_1)} =
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(\sigma(1, a_1^{-1}b_1a_1))\sigma(1, a_1)(1, a_1^{-1}b_1a_1) = (\sigma(1, a_1^{-1}b_1a_1))\sigma(1, b_1a_1) = \\
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$\phi_1 \circ T(b_1) = \phi_1(b_1^{a_1}) = (a_1, 1)\sigma(1, a_1^{-1}b_1a_1) = (1, a_1^{-1}b_1^{-1}a_1b_1a_1)$
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$$

$$
\phi_1 \circ T(b_2) = (1, b_2^{-1} b_1 a_1)
$$
We get the recursion

\[ a_1 \mapsto \sigma(a_1^{-1} b_1^{-1} a_1^{-1} b_1 a_1, b_1 a_1) = \left((a_1^{-1})^T, (b_1 a_1)^T\right) \]

\[ b_1 \mapsto (1, a_1^{-1} b_1^{-1} a_1 b_1 a_1) = (1, a_1^T) \]

\[ b_2 \mapsto (1, b_2^{-1} b_1 a_1) = \left(1, (b_2^{-1} b_1 a_1)^T\right) \]

showing that \( T \cdot \phi_1 = \phi_0 \cdot (T, T) \).
$D \cdot \phi_0$ is

\[
\begin{align*}
a_1 & \mapsto \sigma(a_1^{-1}b_1^{-1}b_2, b_1b_2^{-1}b_1a_1), \\
b_1 & \mapsto (1, a_1), \\
b_2 & \mapsto (1, b_2^{-1}b_1a_1).
\end{align*}
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b_1 &\mapsto (1, a_1), \\
b_2 &\mapsto (1, b_2^{-1}b_1a_1).
\end{align*}
$$

Conjugating the right-hand side by $(1, b_1^{-1}b_2)$, we get

$$
\begin{align*}
a_1 &\mapsto \sigma\left((a_1^{-1})b_1^{-1}b_2, b_1a_1b_1^{-1}b_2\right) = \sigma\left((a_1^{-1})^D, (b_1a_1)^D\right) \\
b_1 &\mapsto \left(1, a_1b_1^{-1}b_2\right) = \left(1, a_1^D\right), \\
b_2 &\mapsto \left(1, b_2^{-1}b_1a_1b_1^{-1}b_2\right) = \left(1, (b_2^{-1}b_1a_1)^D\right),
\end{align*}
$$
$D \cdot \varphi_0$ is

\[
\begin{align*}
  a_1 & \mapsto \sigma (a_1^{-1} b_1^{-1} b_2, b_1 b_2^{-1} b_1 a_1), \\
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\[
\begin{align*}
  a_1 & \mapsto \sigma \left( (a_1^{-1}) b_1^{-1} b_2, b_1 a_1^{b_1^{-1} b_2} \right) = \sigma \left( (a_1^{-1})^D, (b_1 a_1)^D \right) \\
  b_1 & \mapsto \left( 1, a_1^{b_1^{-1} b_2} \right) = \left( 1, a_1^D \right), \\
  b_2 & \mapsto \left( 1, b_2^{-1} b_1 a_1^{b_1^{-1} b_2} \right) = \left( 1, (b_2^{-1} b_1 a_1)^D \right),
\end{align*}
\]

hence $D \cdot \varphi_0 = \varphi_0 \cdot (D, D b_2^{-1} b_1)$. 
Similar computations show that $D \cdot \phi_1 = \phi_1 \cdot (a_1^{-1}, b_2^{-1} b_1 a_1)$. 

Let's summarize:

$T \cdot \phi_0 = \phi_1$,  

$T \cdot \phi_1 = \phi_0 \cdot (T, T)$,  

$D \cdot \phi_0 = \phi_0 \cdot (D, D b_1 - 1 b_2)$,  

$D \cdot \phi_1 = \phi_1 \cdot (a_1^{-1}, b_2^{-1} b_1 a_1)$.

Taking the "direct sum" $\phi_0 \oplus \phi_1$, we get the following wreath recursion for the iterated monodromy group of $F$:

$a_1 = \sigma(a_1^{-1}, b_1 a_1, 1, a_1^{-1} b_1 a_1)$,  

$b_1 = (1, a_1, a_1, 1)$,  

$a_2 = \sigma(a_1^{-1}, b_2 a_2, a_1^{-1}, b_2 a_2)$,  

$b_2 = (1, a_2, 1, a_2)$,

$T = \pi(1, 1, T, T)$,

$D = (D, D b_1 - 1 b_2)$,

where $\sigma = (12)(34)$ and $\pi = (13)(24)$. 

V. Nekrashevych (Texas A&M) Skew product map 2019, November 3 ICERM 15 / 18
Similar computations show that $D \cdot \phi_1 = \phi_1 \cdot (a_1^{-1}, b_2^{-1} b_1 a_1)$. Let’s summarize:

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T \cdot \phi_0 = \phi_1, \quad T \cdot \phi_1 = \phi_0 \cdot (T, T),
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\[
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Similar computations show that \( D \cdot \phi_1 = \phi_1 \cdot (a_1^{-1}, b_2^{-1} b_1 a_1) \). Let’s summarize:

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Taking the “direct sum” \( \phi_0 \oplus \phi_1 \), we get the following wreath recursion for the iterated monodromy group of \( F \):

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D = (D, Db_2^{-1} b_1, a_1^{-1}, a_2),
\]

where \( \sigma = (12)(34) \) and \( \pi = (13)(24) \).
The limit space?

If a map is locally expanding, then its iterated monodromy group is \textit{contracting} in the sense that the word lengths $\| \cdot \|$ of the coordinates of $\phi^n(g)$ are not more than $\lambda \| g \| + C$ for some constants $\lambda \in (0, 1)$, $n$, and $C$.
The limit space?

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In our case we have

$$
(a_1 a_2^{-1})^N \mapsto \left( \left( b_1 a_1 a_2^{-1} b_2^{-1} \right)^N, \left( a_1^{-1} a_2 \right)^N, \left( a_1^{-1} b_1 a_1 a_2^{-1} b_2^{-1} \right)^N, a_2^N \right),
$$

which shows that the wreath recursion is not contracting.
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which shows that the wreath recursion is not contracting. Moreover, the iterated monodromy group (i.e., the smallest quotient compatible with the recursion) is not contracting, since $a_2$ is of infinite order.
If a map is locally expanding, then its iterated monodromy group is *contracting* in the sense that the word lengths $\| \cdot \|$ of the coordinates of $\phi^n(g)$ are not more than $\lambda \| g \| + C$ for some constants $\lambda \in (0, 1)$, $n$, and $C$.

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which shows that the wreath recursion is not contracting. Moreover, the iterated monodromy group (i.e., the smallest quotient compatible with the recursion) is not contracting, since $a_2$ is of infinite order. This means that $F$ is not *sub-hyperbolic*: we can not define an orbifold containing the complement of the post-critical set on which $F$ is locally expanding.
If a map is locally expanding or sub-hyperbolic, then the support of the measure of maximal entropy of the map is uniquely determined by the wreath recursion via the construction of the \textit{limit space}.
If a map is locally expanding or sub-hyperbolic, then the support of the measure of maximal entropy of the map is uniquely determined by the wreath recursion via the construction of the limit space. We may try to see what is the support of the measure of maximal entropy for our example...
The frames of the movie should correspond to the connected components of the limit space of the subgroup

\[ a_1 = \sigma(a_1^{-1}, b_1 a_1, 1, a_1^{-1} b_1 a_1), \]
\[ b_1 = (1, a_1, a_1, 1), \]
\[ a_2 = \sigma(a_2^{-1}, b_2 a_2, a_2^{-1}, b_2 a_2), \]
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They are well defined in this case, even though the group is not contracting.
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They are well defined in this case, even though the group is not contracting. One can show that the components corresponding to rational angles not of the form \( \frac{k}{2^n} \) coincide with the corresponding frames of the movie.
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They are well defined in this case, even though the group is not contracting. One can show that the components corresponding to rational angles not of the form \( \frac{k}{2^n} \) coincide with the corresponding frames of the movie (they are Julia sets of some p.c.f. rational functions of one variable).
The frames of the movie should correspond to the connected components of the limit space of the subgroup

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They are well defined in this case, even though the group is not contracting. One can show that the components corresponding to rational angles not of the form $\frac{k}{2^n}$ coincide with the corresponding frames of the movie (they are Julia sets of some p.c.f. rational functions of one variable). It is not true for the angles $\frac{k}{2^n}$.
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They are well defined in this case, even though the group is not contracting. One can show that the components corresponding to rational angles not of the form \( \frac{k}{2^n} \) coincide with the corresponding frames of the movie (they are Julia sets of some p.c.f. rational functions of one variable). It is \textit{not} true for the angles \( \frac{k}{2^n} \). What about the irrational angles?